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Covering Spaces and the Galois Theory of Commutative Banach Algebras

WILLIAM R. ZAME*

Department of Mathematics, State University of New York at Buffalo, Buffalo, New York 14214

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The principal result of this paper is that there is a bijective (functorial) correspondence between the projective separable extensions of a comutative Banach algebra A and the finite covering spaces of its maximal ideal space $M(A)$. As a consequence, a full Galois theory for commutative Banach algebras is developed which is analogous to the (unramified) Galois theory of function fields on compact Riemann surfaces. In case $M(A)$ is a reasonably "nice" space, its profinite fundamental group is identified as the automorphism group of the separable closure of A.

I. Introduction

One of the recurring themes in the study of commutative Banach algebras is the attempt to understand the relationship between the algebraic structure of a commutative Banach algebra A and the topological structure of its maximal ideal space $M(A)$. The seminal result in this direction is the Shilov idempotent theorem [23], which identifies the zeroth Cech cohomology group $H^0(M(A), \mathbb{Z})$ with the additive subgroup of A generated by the idempotents. In the same vein, Arens [1] and Royden [22] have showed that $H^1(M(A), \mathbb{Z})$ may be identified with $A^{-1}/\exp A$, the quotient of the (multiplicative) group of invertible elements of A by the exponential subgroup. More recently, Forster [10] showed how to identify $H^2(M(A), \mathbb{Z})$ with Pic(A), the group of rank one projective modules over A. For a more extensive discussion and bibliography, we refer the reader to the paper by Taylor [24].

The motivation for the present work was a desire to understand a more "primitive" topological invariant, namely the fundamental group $\pi_1(M(A))$. This is of an essentially different nature from the results cited above, since

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the fundamental group depends covariantly on the space involved rather than contravariantly as with cohomology. Moreover, homotopy equivalence of maps with range $M(A)$ does not seem as amenable to Banach algebra techniques as does homotopy equivalence of maps with domain $M(A)$. (The Cech cohomology results, as well as many others cited in [24], can be obtained by representing the groups in question as homotopy classes of mappings from $M(A)$ into an appropriate classifying space, and then appealing to work of Grauert on holomorphic homotopy classes; see [24] for details.) To circumvent these difficulties, we are led to study $\pi_1(M(A))$ indirectly, through the covering spaces of $M(A)$. In order to understand the way in which the structure of covering spaces of $M(A)$ may be related to the structure of A, it is useful to recall some ideas from algebraic function theory.

Let X, Y be compact (connected) Riemann surfaces and let $Y \to X$ be a finite (unramified) holomorphic covering map. We may then view the field $\mathscr{M}(Y)$ of meromorphic functions on Y as a finite (unramified) separable extension of the field $\mathscr{M}(X)$ of meromorphic functions on X. The covering transformations of Y over X then correspond to the automorphisms of $\mathscr{M}(Y)$ which fix $\mathscr{M}(X)$. In particular, Y is a regular covering space of X if and only if $\mathscr{M}(Y)$ is a Galois extension of $\mathscr{M}(X)$. Moreover, every finite, unramified separable extension of $\mathscr{M}(X)$ arises as $\mathscr{M}(Y)$ for some Riemann surface Y covering X. We may thus equate the finite unramified Galois theory of $\mathscr{M}(X)$ with the finite covering theory of X. In this paper, we are going to carry out the same program for commutative Banach algebras.

The fundamental result of this paper (Theorem 3.1) is that the finite covering spaces of $M(A)$ are in one-to-one correspondence with the finitely generated, projective, separable, faithful algebras over A. (These are the appropriate ring-theoretic analogs of unramified field extensions.) Using this basic result, we develop in the succeeding sections a full (finite) Galois theory for commutative Banach algebras. In particular, we determine (Theorem 4.1) the Galois extensions of A with a given finite group of automorphisms. They are in one-to-one correspondence with the principal bundles over $M(A)$, and the set of principal bundles may, in turn, be identified with a Cech cohomology set. In case the given group is abelian, the set of Galois extensions, the set of principal bundles, and the cohomology set all admit the structures of abelian groups, and the one-to-one correspondence is an isomorphism (Theorem 4.2). In case the given group is cyclic, we obtain (Theorem 5.1) a three-term exact sequence which connects the group of Galois extensions with the groups $A^{-1}/\exp A$ and $\mathrm{Pic}(A)$. As a consequence, we are able to determine (in purely topological terms) when a given extension arises by adjoining roots of an element of A (Theorem 5.2).

It turns out that out methods do not allow us to determine the fundamental group $\pi_1(M(A))$ for a general algebra A. The difficulty is familiar from algebraic function theory: the universal cover of a compact space need not be a finite cover. We can, however, determine a reasonable substitute, the "profinite fundamental group" (Theorem 6.1).

There is a substantial body of work on the subject of Galois theory for commutative Banach algebras. The complete structure theory for $C(X)$, the algebra of continuous functions on a compact space X, was worked out by Childs [5] and Wajnryb [25], while Magid [20] was able to obtain the complete theory for semi-simple regular Banach algebras. (Of course these cases are very much simpler than the general case.) Magid also raised the question of whether his results were valid for general commutative Banach algebras. The paper by Craw [7] details the structure of Galois extensions with an abelian group of automorphisms, although he does not obtain the full strength of Theorem 4.2 and does not treat cyclic extensions at all. Surprisingly, the present paper has relatively little in common with [7]; perhaps that is because the techniques used here are quite different (and more direct), and lead naturally in other directions.

There is also a large body of work peripherally related to the present paper, i.e., the work of Brown [4], Lin–Zjuzin [17], Lindberg [18], and others on Arens–Hoffman extensions of commutative Banach algebras. The essential difference is that the Galois extensions we study need not arise as splitting algebras of polynomials (see Section 7). This in fact exemplifies the difference between the Galois theory of rings and the Galois theory of fields, and gives our work a somewhat different character from work in algebraic function theory.

Portions of this work were announced in [26] and presented at the Annual Meeting of the American Mathematical Society (St. Louis, January, 1977). Portions of this work were also presented at the Workshop on Continuous Lattices and Topological Algebra (Tulane University, April, 1977) and the Conference on Several Complex Variables (Princeton University, April, 1979). The author is grateful for the invitations to speak at these conferences.

II. Banach Algebras and Holomorphic Functions

In this section we collect some necessary information about commutative Banach algebras and holomorphic functions on infinite-dimensional spaces.

All commutative Banach algebras will be assumed to have an identity and all homomorphisms will be assumed to be continuous and preserve the identity. When we speak of the category of commutative Banach algebras, we mean thus the category whose objects are commutative Banach algebras with identity, and whose morphisms are continuous homomorphisms which preserve the identity.

Throughout the remainder of this paper, we let A be a commutative Banach algebra (with identity) and let A^* denote its linear dual space. By the *maximal ideal space* (*spectrum*) of A we mean the compact set $M(A) \subset A^*$ of nonzero homomorphisms of A to $\mathbb{C}$. The elements of $M(A)$ may be identified with the maximal ideals of A; we will sometimes make this identification without further comment. If $a \in A$, then the *Gelfand transform* of a is the continuous map $\hat{a}: M(A) \to \mathbb{C}$ given by $\hat{a}(\gamma) = \gamma(a)$.

If U is an open subset of A^* and f is a continuous, complex-valued function on U, we say that f is *holomorphic* if it has a complex-linear Frechét derivative at each point. This is equivalent to the assertion that $f|(U \cap E)$ is holomorphic, in the usual sense, for each finite-dimensional affine subspace E of A^*. If $L \subset A^*$ is a closed subspace of finite codimension and $\pi_L : A^* \to A^*/L$ is the quotient map, we will say that U is an *L-set* if $U = \pi_L^{-1}\pi_L(U)$ (note that $\pi_L(U)$ is open because π_L is a surjective linear mapping onto a finite-dimensional space). This is evidently equivalent to the assertion that U contains every translate of L which it meets. If $\pi_L(U)$ is a polynomial polyhedron in the finite-dimensional space A^*/L, we will call U and *L-polynomial polyhedron*. If U is an L-set, we say that f is an *L-homomorphic function* if there is a holomorphic function f_L on $\pi_L(U)$ such that $f = f_L \circ \pi_L$: this is equivalent to the assertion that f is holomorphic and is constant on each affine translate of L. We denote the algebra of L-holomorphic functions on the L-set U by $\mathcal{O}_L(U)$. Obviously π_L induces an isomorphism of $\mathcal{O}_L(U)$ with the algebra $\mathcal{O}(\mu_L(U))$ of holomorphic functions on $\pi_L(U)$.

It is evident that an L-holomorphic function is holomorphic and not difficult to see that each holomorphic function is locally L-holomorphic (where the choice of L may vary from point to point). The following result shows that, near $M(A)$, much more is true; for a proof and further details we refer to [24].

PROPOSITION 2.1. *Let U be an open subset of A^* which contains $M(A)$ and let f be a holomorphic function on U. Then there is a closed subspace $L \subset A^*$, of finite codimension, and an L-polynomial polyhedron V such that $M(A) \subset V \subset U$ and $f \mid V$ is an L-holomorphic function.*

Denote by $\mathcal{O}(M(A))$ the algebra of germs on $M(A)$ of functions holomorphic in a neighborhood of $M(A)$. Notice that $\mathcal{O}(M(A))$ may be identified with the direct limit $\varinjlim \mathcal{O}_L(U)$, where the limit is taken over all closed subspaces L of A which are of finite codimension and all L-polynomial polyhedra U. In this setting, Craw [6] proved a version of the usual Arens–Calderón [2] functional calculus. If $a \in A$, we may regard a as a continuous function on A^* by setting $a(\chi) = \chi(a)$ for each $\chi \in A^*$; note that $a \mid M(A)$ is just the Gelfand transform $\hat{a}$. Craw's version of the functional calculus then takes the form of

THEOREM 2.2. *There is a homomorphism* $\Theta: \mathcal{O}(M(A)) \to A$ *such that* $\Theta(a) = a$ *for each* $a \in A$ *(in particular,* $\Theta(1) = 1$*).*

Of course, $\mathcal{O}(M(A))$ also admits a topology and Θ is continuous, but that shall not be of concern to us here. For more information, see [6] or [24].

III. COVERING SPACES AND PROJECTIVE SEPARABLE EXTENSIONS

This section contains our principal results. As discussed in the Introduction, our idea is to establish a correspondence between finite covering spaces of $M(A)$ and (certain) finite extensions of A. We begin by recalling the relevant topological and algebraic notions.

Let X be a topological space. By a *covering* of X we mean a pair (Y, p), where Y is a topological space and $p: Y \to X$ is a continuous function, with the property that each point x of X belongs to an open set U for which $p^{-1}(U)$ admits a decomposition $p^{-1}(U) = \bigcup U^j$ into open sets with $p \mid U^j: U^j \to U$ a homeomorphism onto. We will say that U is *evenly covered* by the sets U^j. (Notice that we make no assumptions as to the connectedness or local connectedness of X, Y. This will require some care, since many of the basic results of covering space theory do not apply in this situation.) If no confusion is likely to result, we may suppress reference to the map p. If (Y_1, p_1) and (Y_2, p_2) are coverings of X, a *morphism* from (Y_1, p_1) to (Y_2, p_2) is a map $q: Y_1 \to Y_2$ such that (Y_1, q) is a covering of Y_2 and $p_2 \circ q = p_1$. We will say that the covering (Y, p) of X is *finite* if $p^{-1}(x)$ is a finite set, for each $x \in X$. It is easy to show that, if X is a compact Hausdorff space and (Y, p) is a finite covering, then Y is also a compact Hausdorff space and the number of points in the fiber $p^{-1}(x)$ is a locally constant function of $x \in X$, and in particular, is bounded on X. For information on coverings in our context, we refer to [11, 13].

For T a commutative ring with identity, a *projective separable extension* of T is a commutative ring R which contains T and has the same identity (equivalently: a faithful commutative, unital T-algebra), and which is finitely generated and projective as a module over T and separable as an algebra over T (i.e., $R \otimes_T R$ is projective as a module over R). By a *morphism* of projective separable extensions of T we shall simply mean an identity-preserving homomorphism of T-algebras. Projective separable extensions play a role in the Galois theory of rings analogous to the role played by finite separable extensions in the Galois theory of fields. In particular, an extension ring R of T is a projective separable extension exactly when there is a Galois extension S of T such that R is isomorphic to the fixed ring of some group of automorphisms of S which fix T. (For the definition of Galois extension, see the next section.) For further discussion and for facts concerning projective separable extensions and the Galois theory of rings (which we sometimes use without specific reference), we refer to [8].

In case $T = A$ is a commutative Banach algebra, it was shown by Magid [20] that each projective separable extension R of A admits the structure of a commutative Banach algebra in a unique way so that A is homeomorphically imbedded as a closed subalgebra. Moreover, any A-homomorphism between projective separable extensions of A is automatically continuous. Hence the category of projective separable extensions of A is a subcategory of the category of commutative Banach algebras (of course, it need not be a full subcategory).

If R is a projective separable extension of A, the inclusion $i_R: A \to R$ induces a map (the adjoint) $M(i_R): M(R) \to M(A)$, which was shown by Magid to be a finite covering. Moreover, if R, S are projective separable extensions of A and $\varphi: R \to S$ is an A-homomorphism, then we have a covering map $M(\varphi): M(S) \to M(R)$ which renders Fig. 3.1 commutative.

$$\begin{tikzcd} M(S) \arrow[r, "M(\varphi)"] \arrow[dr, "M(i_S)"'] & M(R) \arrow[d, "M(i_R)"] \\ & M(A) \end{tikzcd}$$

FIGURE 3.1

Thus, the maximal ideal space functor M induces a (contravariant) functor $\bar{M}$ from the category of projective separable extensions of A to the category of finite covering spaces of $M(A)$, given (on objects) by setting $\bar{M}(R) = (M(R), M(i_R))$. Our principal result may now be formulated in the following way. (For category-theoretic terminology, see [19].)

THEOREM 3.1. *For A a commutative Banach algebra, the functor $\bar{M}$ is a contravariant equivalence from the category of projective separable extensions of A to the category of finite covering spaces of $M(A)$. In particular, for each finite covering (Y, p) of $M(A)$, there is a projective separable extension R of A such that $M(R) = Y$ and $M(i_R) = p$.*

Proof. If was shown by Magid [20] that the functor $\bar{M}$ is full and faithful, i.e., $\bar{M}$ is injective and every morphism between objects which are in the range of $\bar{M}$ is actually a morphism which is in the range of $\bar{M}$. To establish the theorem, it then suffices to prove the last statement. To this end, let (Y, p) be a finite covering of $M(A)$. For convenience, we will write $X = M(A)$.

Our first task is to extend the given covering. More precisely, we claim that there are an open neighborhood $\tilde{X}$ of $X = M(A)$ in A^*, a finite covering $(\tilde{Y}, \tilde{p})$ of $\tilde{X}$, and an injection $\sigma: Y \to \tilde{Y}$ such that Fig. 3.2 is commutative.

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & \tilde{Y} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tilde{p}} \\ X & \hookrightarrow & \tilde{X} \end{array}$$

FIGURE 3.2

Let $\{U_i\}$ be a cover of X by relatively open sets, such that each U_i is evenly covered by $\{U_i^j\}$, with $p^{-1}(U_i) = \bigcup_j U_i^j$. Since the topology of A^* is given by pseudometrics and $X = M(A)$ is compact, there is no loss of generality in assuming that the cover $\{U_i\}$ is finite, that U_i and U_k have disjoint closures if U_i and U_k are disjoint, and that for an appropriate choice of pseudometric d, each U_i is a basic open set, i.e., there are points $x_i \in U_i$ and positive numbers ε_i such that, for each i,

$$U_i = \{x \in X : d(x, x_i) < \varepsilon_i\}.$$

Set $\tilde{U}_i = \{z \in A^* : d(z, U_i) < d(z, X \setminus U_i)\}$; then $\tilde{U}_i$ is an open subset of A^* and $\tilde{U}_i = \tilde{U}_i \cap X$. Moreover, the collection $\{\tilde{U}_i\}$ is easily seen to have the same nerve as the collection $\{U_i\}$, i.e., if I is any set of indices, then $\{\tilde{U}_i : i \in I\}$ has a nonempty intersection exactly if $\{U_i : i \in I\}$ has a nonempty intersection. For each i, j, let $\tilde{U}_i^j$ be a copy of $\tilde{U}_i$; since $p \mid U_i^j$ is a homeomorphism onto U_i, we have a natural inclusion of U_i^j into $\tilde{U}_i^j$. Let Y_0 be the disjoint union of the sets $\tilde{U}_i^j$, and let $p_0: Y_0 \to A^*$ be the natural projection. Notice that p_0 maps $\tilde{U}_i^j$ homeomorphically onto $\tilde{U}_i$. Let $\tilde{Y}$ be the quotient of Y_0 by the equivalence relation which identifies the points $y \in \tilde{U}_i^j$ and $z \in \tilde{U}_k^l$ provided that $p_0(y) = p_0(z)$ and U_i^j meets U_k^l, and let $\tilde{p}$: $\tilde{Y} \to A^*$ be the induced mapping. Since we have an inclusion of U_i^j into $\tilde{U}_i^j$, we obtain a mapping σ: $Y \to \tilde{Y}$. Setting $\tilde{X} = \tilde{p}(\tilde{Y}) = \bigcup \tilde{U}_i$, it is easily seen that $\tilde{X}$, $\tilde{Y}$, $\tilde{p}$, σ enjoy the desired properties.

We can now define the desired extension R of A. Since $\tilde{p}$: $\tilde{Y} \to \tilde{X}$ is a local homeomorphism, it defines an analytic structure on $\tilde{Y}$, i.e., a function g: $V \to \mathbb{C}$ is holomorphic, for $V \subset \tilde{Y}$, if each point of V has an open neighborhood V_0 on which $\tilde{p}$ is a homeomorphism and $g \circ (\tilde{p} \mid V_0)^{-1} : \tilde{p}(V_0) \to \mathbb{C}$ is holomorphic (in the sense discussed in Section 2). Let us suppress the mapping σ and regard Y as a compact subset of $\tilde{Y}$. Let $\mathcal{O}(Y)$ be the algebra of germs on Y of functions holomorphic on a neighborhood of Y in $\tilde{Y}$. Let φ: $\mathcal{O}(X) \to \mathcal{O}(Y)$ be given by $\varphi(f) = f \circ \tilde{p}$; then φ is an isomorphism into, so $\mathcal{O}(Y)$ is an extension of $\varphi(\mathcal{O}(X))$, or equivalently, a faithful algebra over $\mathcal{O}(X)$. Since $X = M(A)$, the functional calculus provides a surjective homomorphism $\Theta: \mathcal{O}(X) \to A$; let I be the kernel of Θ, and let J be the ideal in $\mathcal{O}(Y)$ generated by $\varphi(I)$. We set $R = \mathcal{O}(Y)/J$, and assert that R is a projective separable extension of A with $\bar{M}(R) = (M(R), M(i_R)) = (Y, p)$.

We first carry out a construction which will be used repeatedly in the remainder of the proof. Let L be a closed subspace of A^* which is of finite codimension, and let Q be an L-polynomial polyhedron with $X \subset Q \subset \tilde{X}$, so that $Q_L = \pi_L(Q)$ is a polynomial polyhedron in the finite-dimensional space A^*/L. Assume that the collection of sets $\{\pi_L(\tilde{U}_i \cap Q\}$ has the same nerve as the collection $\{\tilde{U}_i\}$. We will abbreviate this situation by saying that the pair (L, Q) is *admissible*. Let us set $V_i = \pi_L(\tilde{U}_i \cap A)$, and let V_i^j be a copy of U_i, one copy for each of the sets $\tilde{U}_i^j$. If we now follow the procedure used above for constructing $\tilde{Y}$ (i.e., form the disjoint union of the V_i^j, divide out by the appropriate equivalence relation, etc.), we obtain a covering $p_L : Y_L \to Q_L$ and a map σ_L $\sigma_L : \tilde{p}^{-1}(Q) \to Y_L$ such that $P_L \circ \sigma_L = \pi_L \circ (\tilde{p} \mid \tilde{p}^{-1}(Q))$. (Of course, Y_L depends on Q as well as on L.) Notice that if $\tilde{U}_i^j$ and $\tilde{U}_k^l$ are disjoint then the sets $\sigma_L(\tilde{U}_i^j \cap \tilde{p}^{-1}(Q))$ and $\sigma_L(\tilde{U}_k^l \cap \tilde{p}^{-1}(Q))$ are also disjoint. Since Q_L is a polynomial polyhedron, it is in particular a Stein manifold, so Y_L admits the structure of a Stein manifold in a unique way so that p_L is holomorphic. Finally, suppose that (K, P) is another admissible pair with $K \subset L$ and $P \subset Q$. Let $\pi_{KL} : A^*/K \to A^*/L$ be the natural projection. The above construction provides a holomorphic map $\sigma_{KL} : Y_K \to Y_L$ so that Fig. 3.3 is commutative.

$$\begin{tikzcd}
 & \tilde{p}^{-1}(P) \arrow[dl, "\sigma_L"'] \arrow[dr, "\sigma_K"] \arrow[dd, "\tilde{p}"'] & \\
Y_L \arrow[dd, "p_L"] & & Y_K \arrow[ll, "\sigma_{KL}"'] \arrow[dd, "p_K"] \\
 & P \arrow[dl] \arrow[dr] & \\
Q_L & & P_K \arrow[ll, "\pi_{KL}"']
\end{tikzcd}$$

FIGURE 3.3

We need to make two further general observations. First of all, because the sets U_i have compact closures and the closures of any pair U_i, U_k are disjoint whenever U_i, U_k are disjoint, admissible pairs (L, Q) do in fact exist, and if (L, Q) is admissible and $K \subset L$, $P \subset Q$ then (K, P) is also admissible. Second, if f is a holomorphic function on a neighborhood of Y in $\tilde{Y}$, then there is an admissible pair (L, Q) and a holomorphic function $\bar{f}$ on Y_L such that $f = \bar{f} \circ \sigma_L$. The proof is essentially identical to the proof of our Proposition 2.1; see [24] for details.

We now return to our analysis of R. The idea is to perform function-theoretic calculations in $\mathcal{O}(Y_L)$ and $\mathcal{O}(Q_L)$, transfer to $\mathcal{O}(Y)$ and $\mathcal{O}(X)$, and then pass to R and A. Since $\mathcal{O}(Y)$ is an extension algebra of $\varphi(\mathcal{O}(X))$ and J

is the ideal of $\mathcal{O}(Y)$ generated by $\varphi(I)$, we see that $R=\mathcal{O}(Y)/J$ is a unital algebra over $A=\mathcal{O}(X)/I$. To see that it is an extension, we must show R to be a faithful algebra. This amounts to showing that every element of $J\cap\varphi(\mathcal{O}(X))$ already belongs to $\varphi(I)$. So let $f\in\mathcal{O}(X)$ and assume that $\varphi(f)\in J$. This amounts to saying that there is a neighborhood W of X in A^* and functions $g_1,\ldots,g_n\in\mathcal{O}(W)$, $h_1,\ldots,h_n\in\mathcal{O}(\tilde{p}^{-1}(W))$ such that the germs of g_i on X belong to I and

$$f\circ\tilde{p}=\sum(g_i\circ\tilde{p})\,h_i \qquad (\text{on } \tilde{p}^{-1}(W)).$$

In view of Proposition 2.1 and the above remarks, we can find an admissible pair (L,Q) with $Q\subset W$, and holomorphic functions $\bar{f},\bar{g}_1,\ldots,\bar{g}_n\in\mathcal{O}(Q_L)$, $\bar{h}_1,\ldots,\bar{h}_n\in\mathcal{O}(Y_L)$ such that $f=\bar{f}\circ\pi_L$, $g_i=\bar{g}_i\circ\pi_L$, $h_i=\bar{h}_i\circ\sigma_L$, and

$$\bar{f}\circ p_L=\sum(\bar{g}_i\circ p_L)\,\bar{h}_i \qquad (\text{on } Y_L).$$

Since p_L is a locally biholomorphic map, we have in particular that for each point $z\in Q_L$, the germ of $\bar{f}$ at z lies in the ideal of ${}_z\mathcal{O}_{Q_L}$ (the stalk at z of the sheaf of holomorphic functions on Q_L) generated by $\bar{g}_1,\ldots,\bar{g}_n$. Since Q_L is a polynomial polyhedron, it is in particular a Stein manifold, so it follows from Cartan's Theorem B that $\bar{f}$ actually lies in the ideal of $\mathcal{O}(Q_L)$ generated by $\bar{g}_1,\ldots,\bar{g}_n$. Hence the function f on Q lies in the ideal in $\mathcal{O}(Q)$ generated by $g_1,\ldots,g_n$. Since the germs of $g_1,\ldots,g_n$ on X are in the ideal I, we also have the germ of f belonging to I, as desired. Hence R is a faithful algebra over A.

We show next that R is a finitely generated projective module over A. Fix an admissible pair (L,Q). Let $\mathcal{O}_{Y_L}$ and $\mathcal{O}_{Q_L}$ be the sheaves of holomorphic functions on Y_L and Q_L, respectively. Since p_L is a finite covering map, we may regard $\mathcal{O}_{Y_L}$ as a locally free sheaf of $\mathcal{O}_{Q_L}$-modules on Q_L. It is an elementary exercise to show that if $f\in\mathcal{O}(Y_L)$ takes distinct values at distinct points in a fiber $p_L^{-1}(z)$, $z\in Q_L$, then the stalk ${}_z\mathcal{O}_{Y_L}$ is generated as a module over ${}_z\mathcal{O}_{Q_L}$, by $1,f,\ldots,f^k$, where $k+1$ is the number of points in the fiber $p_L^{-1}(z)$. Since Y_L is a Stein manifold, we can find a finite number of functions $f_1,\ldots,f_n$ in $\mathcal{O}(Y_L)$ which separate points. In view of the above remark, powers of these functions, which we denote by $f_1,\ldots,f_N$, generate ${}_z\mathcal{O}_{Y_L}$ as a module over ${}_z\mathcal{O}_{Q_L}$, for every $z\in Q_L$. Define a surjective sheaf homomorphism $\mu\colon\mathcal{O}_{Q_L}^N\to\mathcal{O}_{Y_L}$ by setting $\mu(g_1,\ldots,g_N)=\sum f_i\,g_i$. In view of Cartan's Theorem B, the induced section mapping $\mu_*:\mathcal{O}(Q_L)^N\to\mathcal{O}(Y_L)$ is also surjective. Moreover, since $\mathcal{O}_{Y_L}$ is a locally free sheaf, we can also find a sheaf homomorphism $\lambda\colon\mathcal{O}_{Y_L}\to\mathcal{O}_{Q_L}^N$ such that $\mu\circ\lambda=\text{identity}$; hence $\mu_*\circ\lambda_*=\text{identity}$, where $\lambda_*:\mathcal{O}(Y_L)\to\mathcal{O}(Q_L)^N$ is the induced section map. Notice that we have just established that $\mathcal{O}(Y_L)$ is a finitely generated projective module over $\mathcal{O}(Q_L)$.

Now let (K, P) be an admissible pair with $K \subset L$, $P \subset Q$. We obtain a map $\mu_*^K : \mathcal{O}(P_K)^N \to \mathcal{O}(Y_K)$ by setting

$$\mu_*^K(g_1,\dots, g_N) = \sum (f_i \circ \pi_{KL})\, g_i.$$

If we notice that the map $\lambda_* \circ \mu_* : \mathcal{O}(Q_L)^N \to \mathcal{O}(Q_L)^N$ is given by a matrix (h_{ij}) of functions in $\mathcal{O}(Q_L)$, then we can also define a map $E^K : \mathcal{O}(P_K)^N \to \mathcal{O}(P_K)^N$ via the matrix $(h_{ij} \circ \pi_{KL})$. If $\pi_{KL}^{*N} : \mathcal{O}(Q_L)^N \to \mathcal{O}(P_K)^N$ and $\sigma_{KL}^* : \mathcal{O}(Y_L) \to \mathcal{O}(Y_K)$ are the natural maps, then $\sigma_{KL}^* \circ \mu_* = \mu_*^K \circ \pi_{KL}^{*N}$ and $E^K \circ \pi_{KL}^{*N} = \pi_{KL}^{*N} \circ \lambda_* \circ \mu_*$. We want to find a map $\Lambda^K : \mathcal{O}(Y_K) \to \mathcal{O}(P_K)^N$ such that $\Lambda^K \circ \sigma_{KL}^* = \pi_{KL}^{*N} \circ \lambda_*$ and $\Lambda^K \circ \mu_*^K = E^K$. This will lead to the commutative Fig. 3.4.

$$\begin{array}{ccccc}
\mathcal{O}(P_K)^N & \xrightarrow{\mu_*^K} & \mathcal{O}(Y_K) & \xrightarrow{\Lambda^K} & \mathcal{O}(P_K)^N \\
\uparrow{\scriptstyle \pi_{KL}^{*N}} & & \uparrow{\scriptstyle \sigma_{KL}^*} & & \uparrow{\scriptstyle \pi_{KL}^{*N}} \\
\mathcal{O}(Q_L)^N & \xrightarrow{\mu_*} & \mathcal{O}(Y_L) & \xrightarrow{\lambda_*} & \mathcal{O}(Q_L)^N
\end{array}$$

FIGURE 3.4

The existence of Λ^K will follow immediately once we show that μ_*^K is onto and that $\ker(\mu_*^K) \subset \ker(E^K)$.

To see that μ_*^K is onto, we need only observe that, by our construction, the mapping $\sigma_{KL} : Y_K \to Y_L$ is one-to-one on each fiber of p_K. Thus, since the functions $f_1,\dots,f_n$ separate the points of Y_L, the functions $f_1 \circ \sigma_{KL},\dots,f_n \circ \sigma_{KL}$ separate points in each fiber of p_K. That μ_*^K is onto now follows by exactly the same proof that μ_* was onto.

To see the assertion about kernels, notice that at the level of stalks everything is simple. For each $z \in Q_L$, the stalk ${}_z\mathcal{O}_{Y_L}$ is a free ${}_z\mathcal{O}_{Q_L}$-module, so the map ${}_z\lambda : {}_z\mathcal{O}_{Y_L} \to {}_z\mathcal{O}_{Q_L}^N$ can be thought of as determined by a matrix of functions. If $y \in P_L$ with $\pi_{KL}(y) = z$, then we have a map ${}_y\lambda^K : {}_y\mathcal{O}_{Y_K} \to {}_y\mathcal{O}_{P_K}^N$ induced via the matrix obtained by composing with π_{KL}. Now we clearly get ${}_y\lambda^K \circ \mu_*^K = E^K$ (on stalks) since all maps are given by matrices of functions; in particular, if $\mu_*^K(H) = 0$ for some $H \in \mathcal{O}(P_K)^N$, then $E^K(H) = 0$ in ${}_y\mathcal{O}_{P_K}^N$. Since a function is zero exactly when its image in each stalk is zero, it follows that $\ker(\mu_*^K) \subset \ker(E^K)$, as desired. Hence the mapping Λ^K, which renders Fig. 3.4 commutative, does in fact exist. Notice that we have now shown that $\mathcal{O}(Y_K)$ is finitely generated projective $\mathcal{O}(Q_K)$-module in a manner consistent with the way that $\mathcal{O}(Y_L)$ is a finitely generated projective $\mathcal{O}(Q_L)$-module.

Now, since $Y = \varprojlim Y_K$, we may pass to the inverse limit to obtain homomorphisms $F: \mathcal{O}(X)^N \to \mathcal{O}(Y)$ and $G: \mathcal{O}(Y) \to \mathcal{O}(X)^N$ of $\mathcal{O}(X)$-modules such that $F \circ G =$ identity, and $F(g_1,..., g_N) = \sum g_i(f_i \circ \sigma_L)$. In particular, $\mathcal{O}(Y)$ is a finitely generated projective $\mathcal{O}(X)$-module.

Let $q: \mathcal{O}(Y) \to R$ be the quotient map and $\Theta: \mathcal{O}(X) \to A$, the functional calculus map. We may define $\bar{F}: A^N \to R$ by setting $F(a_1,..., a_N) = \sum a_i q(f_i \circ \sigma_L)$; this is clearly a surjection and $\bar{F} \circ \Theta^N = q \circ F$. To obtain a map $\bar{G}: R \to A^N$ such that $\bar{G} \circ q = \Theta^N \circ G$, we need only show that $\ker(q) \subset \ker(\Theta^N \circ G)$. However, $\ker(q) = J$ and it follows exactly as in the proof that F is onto that $F(I^N) = J$. Hence for $h \in \ker(q)$, we can find $H \in I^N$ with $F(H) = h$ (and of course $\Theta^N(H) = 0$). But $G \circ F$ is homomorphism of $\mathcal{O}(X)$-modules and hence automatically preserves I^N, so $\Theta^N \circ G(h) = \Theta^N \circ G \circ F(H) = 0$. Hence the mapping $\bar{G}$ exists. We thus have the following sequence of A modules and A-module homomorphisms:

$$A^N \xrightarrow{\bar{F}} R \xrightarrow{\bar{G}} A^N$$

Since $\bar{F}$ is onto and $\bar{F} \circ \bar{G} =$ identity (because $F \circ G =$ identity), we conclude that R is ndeed a finitely generated projective A-module.

To see that R is a separable A-algebra, it suffices by [8, Theorem 7.1, p. 72] to show that R/mR is a separable A/m-algebra for each maximal ideal m of A. The ideal m is the kernel of a homomorphism $\gamma \in M(A) = X$; let $p^{-1}(\gamma) = \{y_1,..., y_k\} \subset Y$. We may then define a homomorphism $\Gamma: \mathcal{O}(Y) \to \mathbb{C}^k$ by $\Gamma(f) = (f(y_1),..., f(y_k))$. Observe that the homomorphism $\gamma \circ \Theta: \Theta(X) \to \mathbb{C}$ is given by $\gamma \circ \Theta(f) = f(\gamma)$. It is now easy to check, using function-theoretic arguments as above, that Γ induces a surjective homomorphism $\bar{\Gamma}: R \to \mathbb{C}^k$ whose kernel is mR. Hence $R/mR \simeq \mathbb{C}^k$ is indeed a separable algebra over $A/m \simeq \mathbb{C}$, and R is a separable algebra over A.

Having now established that R is a projective separable extension of A, and in particular is a commutative Banach algebra, it remains to show that $M(R) = Y$ and $M(i_R) = p$, where $i_R: A \to R$ is the inclusion. To this end, for each $y \in Y$ let $\delta_y: \mathcal{O}(Y) \to \mathbb{C}$ be the evaluation map, $\delta_y(f) = f(y)$. Notice that the restriction of δ_y to $\varphi(\mathcal{O}(X))$ maps a function $\varphi(g)$ to $g(p(y))$. It follows easily that δ_y induces a homomorphism $\bar{\delta}_y: R \to \mathbb{C}$ and that $M(i_R)(\bar{\delta}_y)$ is the homomorphism $p(y) \in M(A) = X$. Hence we need only show that $\bar{\delta}_y \neq \bar{\delta}_z$ if $y \neq z$ and that every homomorphism in $M(R)$ is a $\bar{\delta}_y$. The first assertion is simple, since we can find an amdmissible pair (K, P) with $\sigma_K(y) \neq \sigma_K(z)$ and (since Y_K is a Stein manifold) can find a function $g \in \mathcal{O}(Y_K)$ with $g(\sigma_K(y)) \neq g(\sigma_K(z))$. But then $\delta_y(g \circ \sigma_K) \neq \delta_z(g \circ \sigma_K)$ so $\bar{\delta}_y$ and $\bar{\delta}_z$ differ on the image of $g \circ \sigma_K$ in R. To see that every homomorphism of R into $\mathbb{C}$ is of this form, suppose that $\beta: R \to \mathbb{C}$ were a (nonzero) homomorphism, $\beta \neq \bar{\delta}_y$ for any $y \in Y$. Then $\beta \circ q \neq \delta_y$ for any $y \in Y$, where $q: \mathcal{O}(Y) \to R$ is the quotient map. It follows that we may find functions $f_1,..., f_l$ in $\mathcal{O}(Y)$ which have no common zero on Y but which all lie in the kernel of $\beta \circ q$. Passing to an appropriate Y_K and using Steinness of Y_K once again, we may find functions $g_1, ..., g_l$ in $\mathscr{O}(Y)$ such that $\sum f_i g_i = 1$. But then

$$1 = \beta \circ q \left(\sum f_i g_i\right) = \sum \beta \circ q(f_i) \beta \circ q(g_i) = 0,$$

and this contradiction establishes that $\beta = \tilde{\delta}_y$ for some y, as desired. The proof is now complete.

IV. Principal Bundles and Galois Extensions

If R is a projective separable extension of A, it follows immediately from Theorem 3.1 that the automorphisms of R which fix A are in one-to-one correspondence with the homeomorphisms of $M(R)$ which commute with the covering map $M(R) \to M(A)$. Hence this covering is regular exactly when R admits "enough" automorphisms. In order to formalize this idea, we introduce some further notions from ring theory.

Let T be a commutative ring with identity and let G be a finite group. By a *Galois extension* of T with group G we mean a commutative extension ring R of T together with an inclusion of G in the group $\mathrm{Aut}(R)$ of automorphisms of R which fix T, such that:

(i) the fixed ring of G, namely $\mathrm{Fix}(G) = \{r \in R : \sigma(r) = r, \text{ all } \sigma \in G\}$, is just T;

(ii) for every maximal ideal m of R and every $\sigma \in G$, $\sigma \neq 1$, there is an element $r \in R$ such that $(r - \sigma(r)) \notin m$.

(If both T, R are fields, this reduces to the usual notion of Galois extension.) We denote the set of Galois extensions of T with group G by $\mathrm{Ext}(T, G)$. If $T \to S$ is a ring homomorphism, we may regard S as a T-algebra via this homomorphism. Then, given $R \in \mathrm{Ext}(T, G)$, the tensor product $R \otimes_T S$ is easily seen to be a Galois extension of S with group G. Thus $\mathrm{Ext}(\cdot, G)$ may be regarded a a functor fro the category of commutative rings to the category of sets.

We wish to emphasize several places where the Galois theory of rings differs from the Galois theory of fields. The first is that the set $\mathrm{Ext}(T, G)$ always contains at least one element, namely the algebra T^G of all functions from G to T; this is in marked contrast to the situation for fields. The same example shows that, for $R \in \mathrm{Ext}(T, G)$, the full group $\mathrm{Aut}(R)$ of automorphisms of R which fix T may be strictly larger than G (however, the two groups will coincide if all the idempotents of R already belong to T). Finally, we note that we agree to regard elements R, S of $\mathrm{Ext}(T, G)$ as equivalent if there is an isomorphism of R with S which is the identity on T

and respects the action of G. In particular, a different choice of the action of G on a given algebra may lead to an inequivalent Galois extension. For further discussion, we refer again to [8].

It is known that a Galois extension is a projective separable extension. Hence, if A is a commutative Banach algebra and $R \in \mathrm{Ext}(A, G)$, then R admits the structure of a commutative Banach algebra in a unique way which respects the structure of A, and all automorphisms of R which fix A are continuous. Thus the notion of Galois extension does not lead us out of the category of commutative Banach algebras.

We also need to recall some topological facts. If X is a compact space and G is a finite group, then a principal G-bundle over X is a compact space Y together with a free action of G on Y with orbit space X. We regard two principal bundles over X as equivalent if there is a homeomorphism between them which respects the action of G and induces the identity mapping on the orbit space X. Note that a principal G-bundle is a covering space, but that inequivalent bundles may be equivalent as covering spaces. The (equivalence classes of) principal G-bundles over X are in one-to-one correspondence with the elements of the Cech cohomology set $H^1(X, G)$, via the correspondence which identifies a principal bundle with the 1-cocycle of its transition functions. We will treat elements of $H^1(X, G)$ interchangeably as cohomology classs or principal bundles, according to our purpose. Notice that $H^1(\cdot, G)$ is functor from compact spaces to sets. (See [16] for more details.) We can now formalize the connection between Galois extensions and principal bundles.

THEOREM 4.1. *Let G be a finite group. Then the functors* $\mathrm{Ext}(\cdot, G)$ *and* $H^1(M(\cdot), G)$, *from the category of commutative Banach algebras to the category of sets, are naturally equivalent. In particular, for each commutative Banach algebra A, there is a bijection between the sets* $\mathrm{Ext}(A, G)$ *and* $H^1(M(A), G)$.

Proof. For each commutative Banach algebra A, we define a map

$$\mu_A : \mathrm{Ext}(A, G) \to H^1(M(A), G)$$

as follows. For $R \in \mathrm{Ext}(A, G)$, it follows from Theorem 3.1 that the action of G on R induces an action of G on $M(R)$. From condition (ii) of the definition of Galois extension this is a free action and from condition (i) $M(A)$ is the orbit space of this action. We can thus define $\mu_A(R)$ to be the principal G-bundle $M(R)$.

We wish first of all to show that $\{\mu_A\}$ is a natural transformation, so let $\alpha: A \to B$ be a homomorphism. We wish to check the commutativity of Fig. 4.1.

$$\begin{tikzcd}
\operatorname{Ext}(A, G) \arrow[r, "\operatorname{Ext}(\alpha, G)"] \arrow[d, "\mu_A"] & \operatorname{Ext}(B, G) \arrow[d, "\mu_B"] \\
H^1(M(A), G) \arrow[r, "H^1(M(\alpha), G)"] & H^1(M(B), G)
\end{tikzcd}$$

FIGURE 4.1

To this end, let $R \in \operatorname{Ext}(A, G)$. Then $\operatorname{Ext}(\alpha, G)(R)$ is the algebra $R \otimes_A B$, and $\mu_B(R \otimes_A B) = M(R \otimes_A B)$, thought of as a principal G-bundle over $M(B)$. We now have the commutative Fig. 4.2.

$$\begin{tikzcd}
M(R) \arrow[d] & M(R \otimes_A B) \arrow[l] \arrow[d] \\
M(A) & M(B) \arrow[l, "M(\alpha)"']
\end{tikzcd}$$

FIGURE 4.2

A simple computation shows that we may identify $M(R \otimes_A B)$ with the relative product,

$$M(R) \times_{M(A)} (B) = \{(\varphi, \psi) \in M(R) \times M(B) \colon \varphi|_A = \psi \circ \alpha\},$$

so that the above Fig. 4.2 is in fact a pull-back. (The computation is simple because $R \otimes_A B$, being a projective separable extension of B, is already a Banach algebra, so we do not need to worry about completing the tensor product.) However, this is precisely the defining property of the image under $H^1(M(\alpha), G)$ of the G-bundle $M(R)$ in $H^1(M(A), G)$. Hence $\mu_B \circ \operatorname{Ext}(\alpha, G)(R) = H^1(M)(\alpha), G) \circ \mu_A(R)$, and the Fig. 4.1 is indeed commutative, so that $\{\mu_A\}$ is a natural transformation.

Now we need to show that μ_A is a bijection for each A. Suppose first that Y is a principal G-bundle over $M(A)$. Then, in particular, $Y \to M(A)$ is a finite covering so by Theorem 3.1 there is a projective separable extension R of A with $M(R) = Y$, and the action of G on Y induces an action of G as automorphisms of R fixing A. To see that R is a Galois extension of A with group G, we need to verify two conditions. The second condition is easy, for if m is a maximal ideal of R, and σ is an element of G different from the identity then $\sigma(m) \neq m$ (because the action of σ on $M(R)$ is free). Hence there is an element r of R whose Gelfand transform distinguishes $\sigma(m)$ from m, i.e., $\hat{r}(m) - \hat{r}(\sigma(m)) \neq 0$. But $\hat{r}(\sigma(m) = \sigma(r)\hat{\ }(m)$ so that $(r - \sigma(r))\hat{\ }(m) \neq 0$, i.e., $(r - \sigma(r)) \notin m$. To verify condition (i), note that $\operatorname{Fix}(G)$ is a subalgebra of R which contains A, and that, by the above, R is a

Galois extension of $\mathrm{Fix}(G)$ with group G. Hence $M(\mathrm{Fix}(G)) = M(R)/G = Y/G = M(A)$. On the other hand, $\mathrm{Fix}(G)$ is a projective separable extension of A, so it follows, again by Theorem 3.1, that $\mathrm{Fix}(G) = A$, so that R is indeed a Galois extension of A with group G. Hence μ_A is a surjection.

Finally, suppose that $R, S \in \mathrm{Ext}(A, G)$ and that $\mu_A(R) = \mu_A(S)$. Then $M(R)$ and $M(S)$ are equivalent as G-bundles over $M(A)$, i.e., there is a homeomorphism $M(R) \to M(S)$ of spaces over $M(A)$ which preserves the action of G. It follows from Theorem 3.1 that there is an isomorphism $S \to R$ of algebras over A which preserves the action of G. Hence $R = S$, so that μ_A is an injection. This completes the proof.

If we restrict attention to finite abelian groups, then $\mathrm{Ext}(A, G)$ depends functorially on G, in the following way (for details, see [15]). Let $\beta\colon G \to H$ be a homomorphism of finite abelian groups and define a group homomorphism $\gamma\colon G \times H \to H$ by $\gamma(g, h) = \beta(g)\,h$; let K be the kernel of γ. For $R \in \mathrm{Ext}(A, G)$, the tensor product $R \otimes_A A^H$ is a Galois extension of A with group $G \times H$; the ring $S = \mathrm{Fix}(K)$ is then a Galois extension of A with group $(G \times H)/K = H$. Thus we may define

$$\mathrm{Ext}(A, \beta)\colon \mathrm{Ext}(A, G) \to \mathrm{Ext}(A, H),$$

by sending R to S as above.

If we apply this construction to the multiplication map $G \times G \to G$, and identify $\mathrm{Ext}(A, G \times G)$ with $\mathrm{Ext}(A, G) \times \mathrm{Ext}(A, G)$, we obtain an abelian group structure on $\mathrm{Ext}(A, G)$. It may be seen that the map $\mathrm{Ext}(A, \beta)$ given above is in fact a homomorphism. Moreover, if $\alpha\colon A \to B$ is a homomorphism, then Fig. 4.3 is commutative.

$$\begin{tikzcd}
\mathrm{Ext}(A, G) \arrow[r, "\mathrm{Ext}(A,\beta)"] \arrow[d, "\mathrm{Ext}(\alpha,G)"] & \mathrm{Ext}(A, H) \arrow[d, "\mathrm{Ext}(\alpha,H)"] \\
\mathrm{Ext}(B, G) \arrow[r, "\mathrm{Ext}(B,\beta)"] & \mathrm{Ext}(B, H)
\end{tikzcd}$$

FIGURE 4.3

Hence $\mathrm{Ext}(\cdot, \cdot)$ is a bifunctor from the category of commutative Banach algebras crossed with the category of finite abelian groups into the category of Abelian groups. Cech cohomology $H^1(M(\cdot), \cdot)$ is another such bifunctor. Our next result asserts that they are equivalent.

THEOREM 4.2. *The bifunctors* $\mathrm{Ext}(\cdot, \cdot)$ *and* $H^1(M(\cdot), \cdot)$ *are naturally equivalent. In particular, for each commutative Banach algebra A and each finite abelian group G, we have an isomorphism of* $\mathrm{Ext}(A, G)$ *with* $H^1(M(A), G)$.

Proof. The proof is very similar in spirit to the proof of Theorem 4.1, so we will give only a sketch. If A is a commutative Banach algebra and G is a finite abelian group, we let

$$\mu_{(A,G)}: \operatorname{Ext}(A, G) \to H^1(M(A), G),$$

be the map μ_A defined in the proof of Theorem 4.1. We check first that this defines a natural transformation of set-valued functors.

Given homomorphisms $\alpha: A \to B$, $\beta: G \to H$, it suffices to check that Fig. 4.4 commutes.

$$\begin{tikzcd}
\operatorname{Ext}(A, G) \arrow[r, "\mu_{(A,G)}"] \arrow[d, "\operatorname{Ext}(\alpha, G)"] & H^1(M(A), G) \arrow[d, "H^1(M(\alpha), G)"] \\
\operatorname{Ext}(B, G) \arrow[r, "\mu_{(B,G)}"] \arrow[d, "\operatorname{Ext}(B', \beta)"] & H^1(M(G), G) \arrow[d, "H^1(M(B), \beta)"] \\
\operatorname{Ext}(B, H) \arrow[r, "\mu_{(B,H)}"] & H^1(M(B), H)
\end{tikzcd}$$

FIGURE 4.4

Now, the top square commutes by Theorem 4.1, so we need only check the bottom square. Let $R \in \operatorname{Ext}(B, G)$, define $\gamma: G \times H \to H$ by $\gamma(g, h) = \beta(g)\, h$, let K be the kernel of γ and let S be the subring of $R \otimes_B B^H$ which is fixed by K, so that S is the image of R in $\operatorname{Ext}(B, H)$. The H-bundle $M(S) = \mu_{(B,H)}(S)$ over $M(B)$ is then obtained from $M(R \otimes_B B^H) = M(R) \times H$ by dividing out by the action of K and observing that $G \times H/K = H$. On the other hand, $\mu_{(B,G)}(R)$ is just the G-bundle $M(R)$ over $M(B)$, and in order to compute its image in $H^1(M(B), H)$, we form the product $M(R) \times H$, considered as a $G \times H$-bundle over $M(B)$, and divide out by the action of K. Since the end results of these two processes are the same, it follows that

$$\mu_{(B,H)} \circ \operatorname{Ext}(B, \beta)(R) = H^1(M(B), \beta) \circ \mu_{(B,G)}(R),$$

as desired. Hence $\{\mu_{(A,G)}\}$ is indeed a natural tranformation of set-valued functors.

In view of Theorem 4.1, each $\mu_{(A,G)}$ is a bijection; we need only show now that it is a group homomorphism. This follows immediately from functoriality and naturality if we consider the multiplication map $G \times G \to G$ and identify $\operatorname{Ext}(A, G \times G)$ with $\operatorname{Ext}(A, G) \times \operatorname{Ext}(A, G)$ and $H^1(M(A), G \times G)$ with $H^1(M(A), G) \times H^1(M(A), G)$. This completes the proof.

V. Cyclic Extensions

In the previous section we saw that, for G a finite abelian group, we may identify the groups $\mathrm{Ext}(A, G)$ and $H^1(M(A), G)$. In case G is cyclic, we can say quite a bit more.

For n a positive integer, we let $\mathbb{Z}_n$ be the cyclic group of order n, which we identify with the group of complex nth roots of unity. The short exact sequence,

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{q} \mathbb{Z}_n \to 0,$$

gives rise to a long exact cohomology sequence, of which the relevant terms are

$$H^1(M(A), \mathbb{Z}) \xrightarrow{q_*} H^1(M(A), \mathbb{Z}_n) \xrightarrow{\delta} H^2(M(A), \mathbb{Z}).$$

Each of these groups has an interpretation in terms of A. The Arens–Royden theorem [1, 22] provides an isomorphism ξ of $A^{-1}/\exp A$ (the quotient of the group of invertible elements of A by the exponential subgroup) with $H^1(M(A), \mathbb{Z})$. Theorem 4.2 provides an isomorphism $\mu = \mu_{(A, \mathbb{Z}_n)}$ of $\mathrm{Ext}(A, \mathbb{Z}_n)$ with $H^1(M(A), \mathbb{Z}_n)$. And a theorem of Forster [10] provides an isomorphism λ of $\mathrm{Pic}(A)$ (the group of rank one projective modules over A) with $H^2(M(A), \mathbb{Z})$. The following result establishes a link between $A^{-1}/\exp A$, $\mathrm{Ext}(A, \mathbb{Z}_n)$ and $\mathrm{Pic}(A)$.

Theorem 5.1. *There are natural homomorphisms α, β so that Fig.* 5.1 *is commutative.*

$$\begin{array}{ccccc}
A^{-1}/\exp A & \xrightarrow{\alpha} & \mathrm{Ext}(A, \mathbb{Z}_n) & \xrightarrow{\beta} & \mathrm{Pic}(A) \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \lambda} \\
H^1(M(A), \mathbb{Z}) & \xrightarrow{q_*} & H^1(M(A), \mathbb{Z}_n) & \xrightarrow{\delta} & H^2(M(A), \mathbb{Z})
\end{array}$$

Figure 5.1

Proof. We define α in the following way. Let $f \in A^{-1}$ so that $f \cdot \exp A \in A^{-1}/\exp A$, and form the quotient $A[t]/(t^n - f)$ of the polynomial ring over A by the ideal generated by the monomial $(t^n - f)$. This is an Arens–Hoffman extension of A (see [3], for example). A typical element of $A[t]/(t^n - f)$ has a representation of the form $\sum_{j=0}^{n-1} a_j t^j$. Define an action of $\mathbb{Z}_n$ on $A[t]/(t^n - f)$ by setting

$$\omega \cdot \sum a_j t^j = \sum \omega^j a_j t^j.$$

It follows easily from elementary facts about polynomial rings that this action makes $A[t]/(t^n - f)$ into an element of $\mathrm{Ext}(A, \mathbb{Z}_n)$ which depends only on the class $f \cdot \exp A$ and not on the choice of f; hence we have obtained an element $\alpha(f \cdot \exp A)$ of $\mathrm{Ext}(A, \mathbb{Z}_n)$.

We wish to show that $q_* \circ \xi(f \cdot \exp A) = \mu \circ \alpha(f \cdot \exp A)$; this is not hard to do. It is evident that the principal $\mathbb{Z}_n$-bundle corresponding to $R = \alpha(f \cdot \exp A)$ can be obtained in the following way. Let $\hat{f}: M(A) \to \mathbb{C}$ be the Gelfand transform of f and define a function $Q: M(A) \times \mathbb{C} \to \mathbb{C}$ by setting $Q(x, t) = t^n - \hat{f}(x)$. Then $M(R)$ may be identified with the zero set of Q, while the action of $\mathbb{Z}_n$ is given by $\omega \cdot (x, t) = (x, \omega t)$ for $\omega \in \mathbb{Z}_n$. On the other hand, $\xi(f \cdot \exp A)$ is the homotopy class of $f: M(A) \to \mathbb{C} \setminus \{0\}$, while $q_* \circ (f \cdot \exp A)$ is the $\mathbb{Z}_n$-bundle obtained by pulling back to $M(A)$ along $\hat{f}$ the bundle $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $z \mapsto z^n$. It is evident that these two $\mathbb{Z}_n$-bundles over $M(A)$ are equivalent, so that $q_* \circ \xi(f \cdot \exp A) = \mu \circ \alpha(f \cdot \exp A)$, as desired. Since $f \cdot \exp A$ is arbitrary, we conclude that $q_* \circ \xi = \mu \circ \alpha$; since ξ and μ are isomorphisms, we may also conclude that α is a homomorphism.

To define the homomorphism β, let $R \in \mathrm{Ext}(A, \mathbb{Z}_n)$ and set $\varepsilon = \exp(2\pi i/n) \in \mathbb{Z}_n \subset \mathbb{C}$. To avoid confusion, we will use σ for the automorphism of R given by the action of ε (so that $\sigma(r) = \varepsilon \cdot r$); by εr we mean simply the product of the complex scalar ε with r. For $k = 0, \ldots, n-1$, let E_k be the eigenspace of σ belonging to the eigenvalue ε^k, i.e.,

$$E_k = \{r \in R: \sigma(r) = \varepsilon^k r\},$$

and define a map $\tau_k: R \to R$ by setting

$$\tau_k(r) = n^{-1} \sum_{j=0}^{n-1} \varepsilon^{-jk} \sigma^j(r).$$

Direct computation shows that: each τ_k is an A-module homomorphism, each τ_k is idempotent, the range of τ_k is E_k, and $\sum \tau_k =$ identity. Hence each E_k is an A-module and $R = \bigoplus E_k$. Since R is a finitely generated projective module over A, each E_k is also a finitely generated projective module over A.

We want to see that each E_k has rank one. To this end let m be a maximal ideal of A. The action of $\mathbb{Z}_n$ on R induces an action of $\mathbb{Z}_n$ on R/mR. Since R is a Galois extension of A, R/mR is a Galois extension of $A/m = \mathbb{C}$. It follows that $R/mR \simeq \mathbb{C}^n$. On the other hand, $R/mR = \bigoplus(E_k/mE_k)$, and E_k/mE_k is the eigenspace corresponding to the eigenvalue ε^k for the automorphism of R/mR induced by the automorphism σ of R. Hence each of the subspaces E_k/mE_k is one dimensional; in particular, no E_k is the zero module. Thus, $\mathrm{rank}(E_k) \geqslant 1$ for each k; since $R = \bigoplus E_k$ and $n = \mathrm{rank}(R) = \sum \mathrm{rank}(E_k)$, it follows that each E_k has rank one, as desired.

We may now define $\beta(R) = E_1 \in \mathrm{Pic}(A)$. We leave it to the reader to check that β is well-defined. To see that $\delta \circ \mu(R) = \lambda \circ \beta(R)$, we will compute.

Note first that $H^2(M(A), \mathbb{Z})$ may be identified with the set of complex line bundles over $M(A)$. If Y, an element of $H^1(M(A), \mathbb{Z}_n)$, is viewed as a principal $\mathbb{Z}_n$-bundle, then the line bundle $\delta(Y)$ may be identified as $Y \otimes \mathbb{C}$, i.e., the quotient of $Y \times \mathbb{C}$ by the equivalence relation which identifies the point (y, u) with the point $(\omega \cdot y, \omega^{-1}u)$ for $y \in Y$, $u \in \mathbb{C}$, $w \in \mathbb{Z}_n$. Hence $\delta \circ \mu(R) = Y \otimes \mathbb{C}$.

To compute $\lambda \circ \beta(R)$, we first need to review the construction of a line bundle from a projective module; see [24] for details. If P is a rank one projective module over A, then we can find an integer N and A-module homomorphisms $\varphi: P \to A^N$, $\psi: A^N \to P$ such that $\psi \circ \varphi =$ identity (i.e., P is a summand of A^N). The map $\varphi \circ \psi: A^N \to A^N$ is then an A-module homomorphism which induces a bundle projection,

$$\bar{U}: M(A) \times \mathbb{C}^N \to M(A) \times \mathbb{C}^N,$$

defined in the following way. For $\eta \in M(A)$ a homomorphism, if $(a_1, \ldots, a_N) \in A^N$ and $(\eta, \ldots, \eta)(a_1, \ldots, a_N) = (0, \ldots, 0)$, then $(\eta, \ldots, \eta) \circ \varphi \circ \psi(a_1, \ldots, a_N) = (0, \ldots, 0)$. Hence we may define a linear map $U_\eta: \mathbb{C}^N \to \mathbb{C}^N$ by requiring that $U_\eta \circ (\eta, \ldots, \eta) = (\eta, \ldots, \eta) \circ \varphi \circ \psi$. We set $\bar{U}(\eta, t) = (\eta, U_\eta(t))$ for $\eta \in M(A)$, $t \in \mathbb{C}^N$. The range of $\bar{U}$ is a line bundle over $M(A)$, which we define to be $\lambda(P)$. It can be shown that $\lambda(P)$ is independent of the choice of the integer N and the homomorphisms φ, ψ.

In our situation, we may apply [8, Proposition 1.2, p. 80] to conclude that there are elements $r_1, \ldots, r_N; s_1, \ldots, s_N$ in R such that $\sum r_j s_j = 1$ while $\sum r_j \sigma^k(s_j) = 0$ for $k = 1, 2, \ldots, n-1$. Define maps $\varphi_j: E_1 \to A$ by

$$\varphi_j(e) = \sum_{k=0}^{n-1} \sigma^k(es_j).$$

Now let $\varphi = (\varphi_1, \ldots, \varphi_N): E_1 \to A^N$ and define $\psi: A^N \to E_1$ by

$$\psi(a_1, \ldots, a_N) = \tau_1 \left(\sum_{j=1}^{N} a_j s_j \right).$$

It is easy to see that φ, ψ are A-module homomorphisms and that $\psi \circ \varphi =$ identity, so the above procedure provides a map $\bar{U}$, and $\lambda(E_1) = \lambda \circ \beta(R) =$ range of $\bar{U}$.

We claim that there is a bundle isomorphism of $M(R) \otimes \mathbb{C}$ with the range of $\bar{U}$. (This will establish that $\delta \circ \mu(R) = \lambda \circ \beta(R)$. Since R is arbitrary, we will have $\delta \circ \mu = \lambda \circ \beta$. As before, this will also enable us to conclude that β is a homomorphism, and the proof will be complete.) To this end, we first construct a map $\bar{T}$ from $M(R) \times \mathbb{C}$ into $M(A) \times \mathbb{C}^N$. For $\gamma \in M(R)$ a homomorphism, notice that if $e \in E_1$, $\gamma \in E_1$, and $\gamma(e) = 0$, then for $j = 1, 2, \ldots, N$,

$$\begin{aligned}\gamma\left(\sum_k \sigma^k(es_j)\right) &= \gamma\left(\sum \sigma^k(e)\,\sigma^k(s_j)\right)\\ &= \gamma\left(\sum \varepsilon^k e\sigma^k(s_j)\right)\\ &= \sum \varepsilon^k \gamma(e)\,\gamma\sigma^k(s_j)\\ &= 0,\end{aligned}$$

because e is an eigenvector for σ belonging to the eigenvalue ε. We may now define a linear mapping $T_\gamma: \mathbb{C} \to \mathbb{C}^N$ by requiring $T_\gamma \circ \gamma = (\gamma \mid A, \ldots, \gamma \mid A) \circ \varphi$. We now obtain the desired map $\bar{T}$ by setting $\bar{T}(\gamma, t) = (\gamma \mid A, T_\gamma(t))$. It is easy to check that $\bar{T}(\omega \cdot \gamma, \omega t) = \bar{T}(\gamma, t)$ for $\gamma \in M(R)$, $t \in \mathbb{C}$, $\omega \in \mathbb{Z}_n$, so that $\bar{T}$ induces a map,

$$\tilde{T}: M(R) \otimes \mathbb{C} \to M(A) \times \mathbb{C}^N,$$

which may be seen to be an injective mapping of vector bundles over $M(A)$. To see that $\tilde{T}$ is an isomorphism of the line bundle $M(R) \otimes \mathbb{C}$ with the line bundle which is the range of the bundle projection $\bar{U}$, we need only check that $\bar{U}$ is the identity on the range of $\tilde{T}$. This is an easy computation, again using the fact that E_1 is the ε-eigenspace of σ, and the proof is complete.

From the above result, we can derive a useful corollary. We would like to be able to recognize cyclic Arens–Hoffman extensions, i.e., extensions of the form $A[t]/(t^n - a)$ where a is an invertible element of A. From the above proof, we know that a cyclic Arens–Hoffman extension always admits the structure of a Galois extension with group $\mathbb{Z}_n$, so we may restrict our search to elements of $\mathrm{Ext}(A, \mathbb{Z}_n)$. Of course, given an element of $\mathrm{Ext}(A, \mathbb{Z}_n)$ we might ask whether it is isomorphic to some cyclic Arens–Hoffman extension as a Galois extension or simply as an A-algebra. Pleasantly, the two notions coincide and can be characterized purely in topological terms or purely in terms involving only the module structure and the group action.

THEOREM 5.2. *Let R be an element of* $\mathrm{Ext}(A, \mathbb{Z}_n)$. *The following conditions are equivalent*:

(i) *there is an invertible element $a \in A$ such that R is isomorphic to $A[t]/(t^n - a)$ as an A-algebra,*

(ii) *there is an invertible element $b \in B$ such that R is isomorphic to $A[t]/(t^n - b)$ as a Galois extension with group $\mathbb{Z}_n$,*

(iii) *for* σ *the automorphism of* R *given by the action of* $\varepsilon = \exp(2\pi i/n) \in \mathbb{Z}_n \subset \mathbb{C}$, *the eigenspace* $E_1 = \{r \in R: \sigma(r) = \varepsilon r\}$ *is a free rank one* A*-module,*

(iv) *the line bundle* $M(R) \otimes \mathbb{C}$ *over* $M(A)$ *is trivial,*

(v) *there is a continuous* $\mathbb{Z}_n$*-bundle imbedding of* $M(R)$ *into* $M(A) \times \mathbb{C}$.

Proof. The equivalence of (ii), (iii), and (iv) is immediate from Theorem 5.1. Since (ii) obviously implies (i), and (iv) implies (v) (because $M(R)$ imbeds in $M(R) \otimes \mathbb{C}$ as a $\mathbb{Z}_n$-bundle), it suffices to prove that (i) implies (ii), and (v) implies (iv). To this end, assume that a is an invertible element of A and that R is isomorphic to $A[t]/(t^n - a)$ as an A-algebra. Let r be the element of R which corresponds to t. In the notation of (iii), we see that the automorphism σ must cyclically permute the roots of $t^n - a$, so that $\sigma(r) = \varepsilon^k r$ for some k, $1 \leqslant k \leqslant n-1$, k relatively prime to n. If l is an integer such that kl is congruent to 1 modulo n, it is easy to see that R is isomorphic, as a Galois extension with group $\mathbb{Z}_n$, to the Arens–Hoffman extension $A[t]/(t^n - a^l)$, which is (ii). Finally, if $\Gamma: M(R) \to M(A) \times \mathbb{C}$ is an imbedding of $\mathbb{Z}_n$-bundles, we may define a map $\Gamma': M(R) \times \mathbb{C} \to M(A) \times \mathbb{C}$ by setting $\Gamma'(\gamma, u) = u\Gamma(\gamma)$. It is clear that $\Gamma'(\omega \cdot \gamma, \omega^{-1}u) = \Gamma'(\gamma, u)$ for each $\gamma \in M(R)$, $u \in \mathbb{C}$, $\omega \in \mathbb{Z}_n$. Hence Γ' induces a map of $M(R) \otimes \mathbb{C}$ into $M(A) \times \mathbb{C}$, which is easily seen to be an isomorphism of line bundles, so we have (iv). This completes the proof.

A version of Theorem 5.2 was obtained by Downum [9] under the assumptions that $M(A)$ is a sufficiently well-behaved space so that covering space theory works and $M(R)$ is connected. Downum phrased his results in terms of the existence of a *cyclic primitive* (i.e., an element $r \in R$ such that $1, r, \ldots, r^{n-1}$ generate R as an A-module and $r^n \in A$). This is easily seen to be equvalent to our formulation.

For further discussion concerning Arens–Hoffman extensions, see Section 7. The situation for Stein spaces is discussed in [27].

VI. The Profinite Fundamental Group

As was mentioned in the Introduction, the original motivation for this work was to understand the fundamental group of $M(A)$ in terms of the algebraic structure of A. It is to this problem which we now turn.

Throughout this section, we assume that A is a commutative Banach algebra whose maximal ideal space $M(A)$ is connected, locally connected and semi-locally simply connected. These are the appropriate conditions on $M(A)$ which guarantee that the theory of covering spaces and the fundamental group works properly [13]. In particular, the connected covering spaces of $M(A)$ correspond to the (conjugacy classes of) subgroups of $\pi_1(M(A))$.

We are not in general able to compute $\pi_1(M(A))$ in terms of A. The problem is that our methods restrict us to finite coverings, which correspond to subgroups of $\pi_1(M(A))$ which are of finite index. However, $\pi_1(M(A))$ need not be determined by its subgroups of finite index (and in fact, might have no nontrivial subgroups of finite index). We can, however, compute the profinite fundamental group, which is a familiar substitute from algebraic geometry [21].

We say that a group G is *profinite* if for each element $g \in G$, $g \neq 1$, there is a normal subgroup H of G which is of finite index and does not contain g. (Thus the coset gH is not trivial in the finite group G/H.) To an arbitrary group G, we may assign a profinite group G^{prof} in the following way. Let $\mathscr{F}(G)$ denote the set of normal subgroups of G which are of finite index, ordered by containment. If H_1, H_2 are in $\mathscr{F}(G)$, whith $H_1 \subset H_2$, we have a natural homomorphism $G/H_1 \to G/H_2$. Thus $\{G/H: H \in \mathscr{F}(G)\}$ is an inverse system of groups; we let G^{prof} be its inverse limit. It is easy to see that G^{prof} is a profinite group and that there is a natural homomorphism $G \to G^{\text{prof}}$, which is a monomorphism exactly when G is profinite. Moreover, any homomorphism from G to a finite group factors canonically and uniquely through G^{prof}. (Notice, however, that G need not be isomorphic to G^{prof}, even if G is profinite, and that G^{prof} need not be isomorphic to $(G^{\text{prof}})^{\text{prof}}$.) We refer to $\pi_1(X)^{\text{prof}}$ as the *profinite fundamental group* of the space X.

We need one more notion from ring theory. Recall that the *separable closure* of A is a commutative extension algebra $\Omega(A)$ of A such that:

(i) for every finite set $F \subset \Omega(A)$, there is a projective, separable A-algebra R with $F \subset R \subset \Omega(A)$,

(ii) if S is a projective, separable extension of $\Omega(A)$, then S is isomorphic to a direct sum of copies of $\Omega(A)$.

The separable closure does in fact exist, and is unique up to isomorphism. It may be thought of as the ring-theoretic analog of the maximal unramified algebraic extension of a field (which is in general much smaller than the full algebraic closure). The separable closure of A may be realized as the union (direct limit) of all Galois extensions of A which have no idempotents other than 0 and 1. (Recall that we have assumed $M(A)$ to be connected, so that A itself has no idempotents other than 0 and 1.) We can now give our interpretation of the profinite fundamental group.

THEOREM 6.1. *Under the above assumptions, there is an isomorphism between the profinite fundamental group $\pi_1(M(A))$ and the group* $\operatorname{Aut}(\Omega(A))$ *of all automorphisms of the separable closure of A which fix A.*

Proof. The proof is basically a diagram chase. Write $X = M(A)$ and $G = \pi_1(X)$. For $H \in \mathscr{F}(G)$, let $Y_H \to X$ be the covering space corresponding to H and let R_H be the projective separable extension of A corresponding to Y_H. Covering space theory provides an isomorphism between G/H and the group of covering transformations of Y_H, while Theorem 3.1 provides an isomorphism between the latter group and the group $\mathrm{Aut}(R_H)$ of automorphisms of R_H which fix A. If $K \subset H$ then Y_K covers Y_H and R_K is an extension of R_H. Moreover, every automorphism of R_K which fixes A also leaves R_H invariant (as a set). Hence we have a natural restriction map $\mathrm{Aut}(R_K) \to \mathrm{Aut}(R_H)$. It is easy to check that Fig. 6.1 is commutative, where the vertical arrows are isomorphisms.

$$\begin{tikzcd} G/K \arrow[r] \arrow[d] & G/H \arrow[d] \\ \mathrm{Aut}(R_K) \arrow[r] & \mathrm{Aut}(R_H) \end{tikzcd}$$

FIGURE 6.1

On the other hand, as the gourp H ranges over $\mathscr{F}(G)$, the algebra R_H ranges over all Galois extensions of A which have no idempotents other than 0 and 1. Hence the construction of the separable closure given in [8, Theorem 3.3, p. 103] shows that

$$\begin{aligned} \pi_1(M(A))^{\mathrm{prof}} = G^{\mathrm{prof}} &= \varprojlim \mathrm{Aut}(R_H) \\ &= \mathrm{Aut}(\varinjlim R_H) \\ &= \mathrm{Aut}(\Omega(A)), \end{aligned}$$

which is the desired result.

We note that the groups $\pi_1(M(A))^{\mathrm{prof}}$ and $\mathrm{Aut}(\Omega(A))$ are both compact totally disconnected groups and that $\pi_1(M(\cdot))^{\mathrm{prof}}$ and $\mathrm{Aut}(\Omega(\cdot))$ may be regarded as functors. It is not hard to see that the isomorphism given above in fact defines a natural equivalence between these two functors; we omit the details.

Finally, an argument similar to that above may be used to show that the profinite homology group $H_1(M(A), \mathbb{Z})^{\mathrm{prof}}$ may be identified with $\mathrm{Aut}(\Omega^{\mathrm{ab}}(A))$, where $\Omega^{\mathrm{ab}}(A)$ is the union of all Galois extensions R of A, for which R has no idempotents other than 0 and 1 and $\mathrm{Aut}(R)$ is abelian. Again, the correspondence is a natural equivalence of functors. (The algebra $\Omega^{\mathrm{ab}}(A)$ may be thought of as the ring-theoretic analog of the maximal unramified abelian extension of a field.)

VII. Arens–Hoffman Extensions

We mentioned in the Introduction that the Galois theory developed here differs from that of [4, 17, 18] because there are Galois extensions which are not Arens–Hoffman extensions. We give here two examples to illustrate our point.

Recall first that an Arens–Hoffman extension of A is the quotient $A[t]/(Q(t))$ of the polynimial ring by a monic polynomial [3]. An Arens–Hoffman extension is a projective separable extension exactly when the discriminant of the polynomial $Q(t)$ is invertible [4].

We first give a simple example of a Galois extension which is not an Arens–Hoffman extension. Let Y be the unit sphere in $\mathbb{R}^3$, let $\mathbb{Z}_2 = \{\pm 1\}$ act on Y by multiplication and let X be the orbit space (the real projective plane). Then Y is a principal $\mathbb{Z}_2$-bundle over X and $C(Y)$ is a Galois extension of $C(X)$ with group $\mathbb{Z}_2$. On the other hand, $C(Y)$ is not an Arens–Hoffman extension $C(X)[t]/Q(t))$. For if it were, there would be a function $f \in C(Y)$ such that $C(Y)$ is generated, as a $C(X)$-module, by the nonnegative powers of f. In particular, f would necessarily distinguish every pair of points in a fiber of Y over X, i.e., f would distinguish every pair of antipodal points of the 2-sphere, which contradicts the Borsuk–Ulam theorem.

We now present a more complicated example to show that being an Arens–Hoffman extension is not determined by maximal ideal spaces alone. It was shown in [12] that there is a function algebra A with the following properties (where we write $X = M(A)$):

(a) every polynomial over A whose discriminant is invertible splits over A,

(b) there is a monic polynomial $q(t)$ over $C(X)$ whose discriminant is invertible and which is irreducible over $C(X)$.

Thus $C(X)[t]/q(t))$ is an Arens–Hoffman extension of $C(X)$ and is also a projective separable extension, so may be identified with $C(Y)$ for some finite covering space Y of X. By our Theorem 3.1, there is a projective separable extension R of A with $M(R) = Y$. Since every polynomial over A whose discriminant is invertible splits over A, R could only be an Arens–Hoffman extension of A if Y were disconnected. However, the irreducibility of $q(t)$ guarantees that Y is in fact connected, so that R is not an Arens–Hoffman extension of A even though $C(M(R)) = C(Y)$ is an Arens–Hoffman extension of $C(M(A)) = C(X)$.

For related results on Stein spaces, see [27].

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